

Parameterized and Approximation Algorithms for the Load Coloring Problem

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Abstract

Let c, k be two positive integers and let $G = (V, E)$ be a graph. The (c, k) -Load Coloring Problem (denoted (c, k) -LCP) asks whether there is a c -coloring $\varphi : V \rightarrow [c]$ such that for every $i \in [c]$, there are at least k edges with both endvertices colored i . Gutin and Jones (IPL 2014) studied this problem with $c = 2$. They showed $(2, k)$ -LCP to be fixed parameter tractable (FPT) with parameter k by obtaining a kernel with at most $7k$ vertices. In this paper, we extend the study to any fixed c by giving both a linear-vertex and a linear-edge kernel. In the particular case of $c = 2$, we obtain a kernel with less than $4k$ vertices and less than $8k$ edges. These results imply that for any fixed $c \geq 2$, (c, k) -LCP is FPT and that the optimization version of (c, k) -LCP (where k is to be maximized) has an approximation algorithm with a constant ratio for any fixed $c \geq 2$.

1 Introduction

Given a graph $G = (V, E)$ and an integer k , the 2-LOAD COLORING Problem asks whether there is a coloring $\varphi : V \rightarrow \{1, 2\}$ such that for $i = 1$ and 2 , there are at least k edges with both endvertices colored i . This problem is NP-complete [1], and Gutin and Jones studied its parameterization by k [6]. They proved that 2-LOAD COLORING is fixed-parameter tractable by obtaining a kernel with at most $7k$ vertices. It is natural to extend 2-LOAD COLORING to any number c of colors as follows. Henceforth, for a positive integer p , $[p] = \{1, 2, \dots, p\}$.

Definition 1 ((c, k) -LOAD COLORING). *Given a positive integer c , a nonnegative integer k and graph $G = (V, E)$, the (c, k) -LOAD COLORING Problem asks whether there is a c -coloring $\varphi : V \rightarrow [c]$ such that for every $i \in [c]$, there are at least k edges with both endvertices colored i . We write $G \in (c, k)$ -LCP if such a c -coloring exists.*

Observe first that $G \in (1, k)$ -LCP if and only if $|E(G)| \geq k$. In this paper, we consider (c, k) -LOAD COLORING parameterized by k for every fixed $c \geq 2$. Note that (c, k) -LOAD COLORING is NP-complete for every fixed $c \geq 2$. Indeed, we can reduce $(2, k)$ -LOAD COLORING to (c, k) -LOAD COLORING with $c > 2$ by taking the disjoint union of G with $c - 2$ stars $K_{1,k}$.

We prove that the problem admits a kernel with less than $2ck$ vertices. Thus, for $c = 2$ we improve the kernel result of [6]. To show our result, we introduce reduction rules, which are new even for $c = 2$. We prove that the reduction rules can run in polynomial time and we show that a reduced graph with at least $2ck$ vertices is in (c, k) -LCP.

While there are many parameterized graph problems which admit kernels linear in the number of vertices, usually only problems on classes of sparse graphs admit kernels linear in the number of edges (since in such graphs the number of edges is linear in the number of vertices), see, e.g., [2, 4, 7]. To the best of our knowledge, only trivial $O(k)$ -edge kernels for general graphs have been described in the literature, e.g., the kernel for MAX CUT parameterized by solution size. Thus, our next result is somewhat surprising: (c, k) -LOAD COLORING admits a kernel with $O(k)$ edges for every fixed $c \geq 2$. In fact, $(2, k)$ -LOAD COLORING has a kernel with less than $8k$ edges and for every $c \geq 2$, (c, k) -LOAD COLORING has a kernel with less than $16c^2k - 6ck$ edges.

The optimization version of (c, k) -LOAD COLORING, called the c -LOAD COLORING Problem, is as follows: for a graph G and an integer $c \geq 2$, find the maximum k such that $G \in (c, k)$ -LCP. The above bounds on the number of edges in the kernel lead to approximation algorithms for this optimization problem: a $(4 + \varepsilon)$ -approximation for $c = 2$ and a constant ratio approximation for $c > 2$.

The paper is organized as follows. In Section 2, we provide additional terminology and notation. In Section 3, we show that the problem admits a kernel with less than $2ck$ vertices. In Section 4, we prove an upper bound on the number of edges in a kernel for every $c \geq 2$ and the corresponding approximation result for c -LOAD COLORING. We improve our bound for $c = 2$ in Section 5. The bound implies the approximation ratio of $4 + \varepsilon$ for every $\varepsilon > 0$. We complete the paper with discussions in Section 6.

2 Terminology and Notation

Graphs. For a graph G , $V(G)$ ($E(G)$, respectively) denotes the vertex (edge, respectively) set of G , $\Delta(G)$ denotes the maximum degree of G , n its number of vertices, and m its number of edges. For a vertex x and vertex set X in G , $N(x) = \{y : xy \in E(G)\}$ and $N_X(x) = N(x) \cap X$. For disjoint vertex sets X, Y of G , let $G[X]$ be the subgraph of G induced by X , $E(X) = E(G[X])$ and $E(X, Y) = \{xy \in E(G) : x \in X, y \in Y\}$. A vertex u with degree 0 (1, respectively) is an *isolated vertex* (a *leaf-neighbor of v* , where $uv \in E(G)$, respectively). For a coloring φ , we say that an edge uv is *colored i* if $\varphi(u) = \varphi(v) = i$.

Parameterized complexity. A parameterized problem is a subset $L \subseteq \Sigma^* \times \mathbb{N}$ over a finite alphabet Σ . L is *fixed-parameter tractable* (FPT) if the membership of an instance (x, k) in $\Sigma^* \times \mathbb{N}$ can be decided in time $f(k)|x|^{O(1)}$, where f is a computable function of the parameter k only. A *kernelization* of a parameterized problem L is a polynomial-time algorithm that maps an instance (x, k) to an instance (x', k') , the *kernel*, such that $(x, k) \in L$ if and only if $(x', k') \in L$, $k' \leq g(k)$, and $|x'| \leq g(k)$ for some function g of k only. We call $g(k)$ the *size* of the kernel.

It is well-known that a parameterized problem L is FPT if and only if it is decidable and admits a kernelization. Due to applications, low degree polynomial size kernels are of main interest. Unfortunately, many FPT problems do not have kernels of polynomial size unless the polynomial hierarchy collapses to the third level [3, 4]. For further background and terminology on parameterized complexity we refer the reader to the monographs [3, 4, 5, 8].

3 Bounding Number of Vertices in Kernel

In this section, we show that (c, k) -LOAD COLORING admits a kernel with less than $2ck$ vertices. A matching with $2ck$ vertices suggests that this bound is likely to be optimal.

For $\tau \in \{<, \leq, =, >, \geq\}$ and integer $i \geq 1$, $K_{1,\tau i}$ denotes a star $K_{1,j}$ with $j \tau i$ and $j \geq 1$. For example, $K_{1,\leq p}$ is a star with q edges such that $q \in [p]$. A $K_{1,\tau i}$ -graph is a forest in which every component is a star $K_{1,\tau i}$, and a $K_{1,\tau i}$ -cover of G is a $K_{1,\tau i}$ -subgraph F of G such that $V(F) = V(G)$. We call any $K_{1,\tau i}$ -graph a *star graph* and any $K_{1,\tau i}$ -cover a *star cover*.

We first prove the bound for star graphs with small maximum degree.

Lemma 1. *If G is a $K_{1,<2k}$ -graph with $n \geq 2ck$, then $G \in (c, k)$ -LCP.*

Proof. The idea is to find for each color some induced subgraph with at least k edges and at most $2k$ vertices. If such subgraphs exist, it is possible to color at most $2ck$ vertices of the graph to obtain k edges for each of the c colors. We prove the lemma by induction on c . The base case of $c = 1$ holds since a $K_{1,<2k}$ -graph G with at least $2k$ vertices has at least k edges (observe that a $K_{1,<2k}$ -graph has no isolated vertices).

Observe now that because all components of G are trees, for each one the number of vertices is one more than the number of edges. If there is a component C , with $k \leq |E(C)| < 2k$, color $V(C)$ with the same color. Then we have used $|V(C)| \leq 2k$ vertices. Thus, we may assume that every component has less than k edges and let C_1, C_2, \dots, C_t be the components of G . Let b be the minimum nonnegative integer for which there exists $I \subseteq [t]$ such that $\sum_{i \in I} |E(C_i)| = k + b \geq k$. Since there is no isolated vertex in a star graph, $m \geq n/2 \geq ck$, and thus such a set I exists. Observe that for any $i \in I$, $|E(C_i)| > b$, as otherwise $\sum_{j \in I \setminus \{i\}} |E(C_j)| = k + b - |E(C_i)| \geq k$, a contradiction to the minimality of b . Since every component has less than k edges, $b \leq k - 2$.

For a star (V, E) , the ratio $\frac{|V|}{|E|}$ decreases when $|E|$ increases. Thus, we have $\sum_{j \in I} |V(C_j)| \leq (k+b) \frac{b+2}{b+1}$. But $2k - (k+b) \frac{b+2}{b+1} = \frac{(k-2-b)b}{b+1} \geq 0$, and so $\sum_{j \in I} |V(C_j)| \leq 2k$. We may color the components C_i , $i \in I$, by the same color. Observe that $H = G - V(\bigcup_{i \in I} C_i)$ has at least $2(c-1)k$ vertices and so $H \in (c-1, k)$ -LCP by the induction hypothesis. Thus, $G \in (c, k)$ -LCP. \square

Since $G \in (c, k)$ -LCP whenever G has a subgraph $H \in (c, k)$ -LCP, we have that any graph with $n \geq 2ck$ and a $K_{1, < 2k}$ -cover is in (c, k) -LCP.

We introduce now a family $(O_{i,k})_{i,k \in \mathbb{N}}$ of obstacles.

Definition 2. We call a pair (V_1, V_2) of disjoint vertex sets an obstacle from $O_{i,k}$ if $|V_1| = i$, $N(v) \subseteq V_1$ for all $v \in V_2$, and for every $u \in V_1$ there is a set $V_u \subseteq N_{V_2}(u)$ such that $|V_u| \geq k$ and for every pair u, v of distinct vertices of V_1 , $V_u \cap V_v = \emptyset$.

Note that if v is an isolated vertex, the pair $(\emptyset, \{v\})$ is an obstacle from $O_{0,k}$.

Observe that if an obstacle (V_1, V_2) from $O_{i,k}$ is contained in a graph G , then $G[V_1 \cup V_2] \in (i, k)$ -LCP: color each $u \in V_1$ and V_u with one color. However, $G[V_1 \cup V_2] \notin (i+1, k)$ -LCP. Indeed, every edge in $G[V_1 \cup V_2]$ is incident to at least one of the i vertices in V_1 . Thus, an edge can only be colored with one of $|V_1| = i$ colors. From this observation, we deduce the following set of reduction rules.

Reduction rule $R_{i,k}$. If an instance G for (c, k) -LCP contains an obstacle (V_1, V_2) from $O_{i,k}$, delete all the vertices of $V_1 \cup V_2$ and decrease c by i .

Now we will prove that Rules $R_{i,k}$ are safe and can be applied in time polynomial in n (recall that c is fixed).

Lemma 2. Let G be a graph and G' be the graph obtained from G after applying reduction rule $R_{i,k}$. Then $G \in (c, k)$ -LCP if and only if $G' \in (c-i, k)$ -LCP.

Proof. For a positive integer p , we call a coloring of an instance G of (c, k) -LCP a *good coloring with p colors* if for at least p colors $j \in [c]$, there are at least k edges colored with color j .

If $G' \in (c-i, k)$ -LCP, then $G \in (c, k)$ -LCP, since a good coloring of the obstacle with i colors together with a good coloring of G' with $c-i$ colors gives a good coloring of G with c colors. On the other hand, if $G \in (c, k)$ -LCP, then it has a good coloring with c colors. In this coloring, there are at least $c - |V_1| = c - i$ colors with no edge with endvertices in V_1 . These colors must have their k edges in $E(G - V_1) = E(G')$. Thus $G' \in (c-i, k)$ -LCP. \square

Lemma 3. One can decide whether Rule $R_{i,k}$ is applicable to G in time $O(n^{i+O(1)})$.

Proof. Generate all i -size subsets V_1 of $V(G)$. For each V_1 , construct the set V_2 that includes every vertex outside V_1 whose only neighbors are in V_1 . If $|V_2| \geq ik$, construct the following bipartite graph B : the partite sets of B are V_1' and V_2 , where V_1' contains i copies of every vertex v of V_1 with the same

neighbors as v . Observe that B has a matching covering V'_1 if and only if $R_{i,k}$ can be applied to G for the obstacle (V_1, V_2) . It is not hard to turn the above into an algorithm of runtime $O(n^{i+O(1)})$. \square

We say that a graph is *reduced for (c, k) -LCP* if it is not possible to apply any rule $R_{i,k}$, $i < c$ to the graph.

Lemma 4. *Let G be a reduced graph for (c, k) -LCP and let $G \notin (c, k)$ -LCP. Then G has a $K_{1, \leq \max\{3, k\}}$ -cover.*

Proof. Let G be such a reduced graph. We first show that G has a star cover. Since it is not possible to apply $R_{0,k}$, G has no isolated vertex. By choosing a spanning tree of each component of G , we obtain a forest F . If a tree in F is not a star, it has an edge not incident to a leaf. As long as F contains such an edge, delete it from F . Observe that F becomes a star cover of G . However, the number of leaves in each star of F is only bounded by $\Delta(G)$. We will show that among the possible star covers of G , there exists a $K_{1, \leq \max\{3, k\}}$ -cover.

For each star cover F , we define the F -sequence $(n_{F, \Delta(G)}, n_{F, \Delta(G)-1}, \dots, n_{F, 1})$, where $n_{F, i}$ is the number of stars with exactly i edges, $i \in [\Delta(G)]$. We say a star cover F_1 is *smaller* than a star cover F_2 if and only if the F_1 -sequence is smaller than the F_2 -sequence lexicographically, i.e. there exists some $i \in [\Delta(G)]$ such that $n_{F_1, i} < n_{F_2, i}$ and for every $j > i$, $n_{F_1, j} = n_{F_2, j}$. We select a star cover S which has the lexicographically minimum sequence, that is, for any star cover $F \neq S$ of G , the S -sequence is smaller or equal to the F -sequence. Suppose that $\Delta(S) > \max\{3, k\}$. Let C_i (L_i , respectively) be the set of all the centers (leaves, respectively) of all stars of S isomorphic to $K_{1, i}$. We also define $L_{\geq i} = \cup_{j \geq i} L_j$. We will now prove two claims.

Claim 1 *There is no edge $uv \in E(G) \setminus E(S)$ such that $u \in L_{\geq 3}$ and $v \in L_{\geq 1}$.*

Indeed, suppose there exists one and let x, y be such that $xu \in E(S)$, $yv \in E(S)$. If $v \in L_{\geq 2}$, then by deleting edges xu, yv and adding edge uv , we do not create any isolated vertex but we decrease the size of the stars centered at x and y , and thus we get a smaller star cover than S , a contradiction. Otherwise, v is an endvertex of an independent edge, and by deleting edge xu and adding edge uv , we decrease the size of the star centered at x , and create a star $K_{1, 2}$ centered at v , which still induces a star cover smaller than S , a contradiction.

Claim 2 *Suppose S contains a star isomorphic to $K_{1, i}$ and centered at vertex x , and a star isomorphic to $K_{1, j}$ and centered at vertex y , such that $i - j \geq 2$. There is no path from x to y in which the odd edges are in $E(S)$ and go from a center to a leaf, and the even edges are in $E(G) \setminus E(S)$ and go from a leaf to a center.*

Suppose there exists such a path. Then by deleting the odd edges of the path and adding the even ones, we do not create isolated vertices because x still has leaf-neighbors, y gets a neighbor, every transitional center keeps the same number of leaf-neighbors and the transitional leaves always go to a new center.

This operation only decreases the size of star centered at x by 1 and increases the size of star centered at y by 1, giving us a lexicographically smaller star cover, a contradiction.

Now, let S' be the subgraph of S containing all stars $K_{1,\Delta(S)}$ of S . While there is an edge $uv \in E(G) \setminus E(S)$ such that u is a leaf of S' and $v \in C_{\Delta(S)-1} \setminus S'$, we add the star centered at v to S' . This procedure terminates because $C_{\Delta(S)-1}$ is finite.

Let C' (L' , respectively) be the centers (leaves, respectively) in S' . Assume now there is an edge $uv \in E(G) \setminus E(S)$ such that $u \in L' \subseteq L_{\geq \Delta(S)-1} \subseteq L_{\geq 3}$ and $v \in V(G) \setminus C'$. By Claim 1, $v \notin L_{\geq 1}$. Since $v \notin C_{\Delta(S)} \subseteq C'$ and since the above procedure has terminated, $v \in C_j$ for some j such that $\Delta(S) - j \geq 2$. Now, by construction, there is an alternating path from a vertex in $C_{\Delta(S)}$ to a vertex in C_j of the type described in Claim 2, which is impossible.

So, there is no edge $uv \in E(G) \setminus E(S)$ such that $u \in L'$ and $v \notin C'$. This means that for any $u \in L'$, $N(u) \subseteq C'$. Furthermore, for each $u \in C'$, we can define V_u to be the leaves of the star centered at u , for which we have $|V_u| \geq \Delta(S) - 1 \geq k$. So, (C', L') is an obstacle from $O_{|C'|,k}$. Since G is reduced for (c, k) -LCP, $|C'| \geq c$ and thus $G[S'] \in (|C'|, k)$ -LCP. This implies that $G \in (c, k)$ -LCP, a contradiction. \square

Now we can prove the following:

Theorem 1. *For every fixed c , if G is reduced for (c, k) -LCP and has at least $2ck$ vertices, then $G \in (c, k)$ -LCP. Thus, (c, k) -LOAD COLORING admits a kernel with less than $2ck$ vertices.*

Proof. Observe that for every c , $G \in (c, 0)$ -LCP, and $G \in (c, 1)$ -LCP if and only if G has a matching with at least c edges. Thus, we may assume that $k \geq 2$. By Lemmas 2 and 3, we can map, in polynomial time, any instance (G, c) into an instance (G', c') such that $c' \leq c$ and G' is reduced for (c', k) -LCP. We therefore may assume that G is reduced for (c, k) -LCP. Suppose that $G \notin (c, k)$ -LCP and $n \geq 2ck$. By Lemma 4, G has a $K_{1, \leq \max(3, k)}$ -cover which is a $K_{1, < 2k}$ -cover, since we assumed $k \geq 2$. But then, Lemma 1 implies that $G \in (c, k)$ -LCP, a contradiction. \square

4 Bounding Number of Edges in Kernel

In the previous section, we proved that (c, k) -LOAD COLORING admits a kernel with less than $2ck$ vertices. We would like to bound the number of edges in a kernel for the problem.

Lemma 5. *Let $b(c, k, n) = c^2k + n(c - 1)$. For every integer $i \geq 0$ and bipartite graph G with n vertices, if $m \geq b(2^i, k, n)$ then $G \in (2^i, k)$ -LCP.*

Proof. We will prove the lemma by induction on i . For the base case, observe that any graph with at least $k = b(1, k, n)$ edges is in $(1, k)$ -LCP for every k

and n . We now assume the claim holds for any j smaller than $i + 1$ and want to prove it for $i + 1$. Consider a bipartite graph $G = (A \cup B, E)$ with n vertices such that $G \notin (2^{i+1}, k)$ -LCP. Let $A_1 = A$, $B_1 = B$ and $A_2 = B_2 = \emptyset$. While there exists $u \in B_1$ such that $|E(A, B_2 \cup \{u\})| < b(2^i, k, |A| + |B_2 \cup \{u\}|) + b(2^i, k, |B_2 \cup \{u\}|)$, move u from B_1 to B_2 . So now assume there is no such u . Then, while $|E(A_1, B_1)| \geq b(2^i, k, |A_1| + |B_1|) + |A_1|$ and $|E(A_2, B_1)| < b(2^i, k, |A_2| + |B_1|) + |A_2|$, move an arbitrary vertex from A_1 to A_2 . Since we only move vertices from A_1 to A_2 or from B_1 to B_2 , we always have $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. Eventually, the partition of $A \cup B$ falls into one of two cases:

- $|E(A_1, B_1)| < b(2^i, k, |A_1| + |B_1|) + |A_1|$. If $A_2 = \emptyset$, then $|E(A_2, B_1)| = 0$. Otherwise, let v be the last vertex moved from A_1 to A_2 . Observe that $|E(A_2, B_1)| \leq |E(A_2 \setminus \{v\}, B_1)| + |B_1| < b(2^i, k, |A_2 \setminus \{v\}| + |B_1|) + |A_2 \setminus \{v\}| + |B_1|$. In both cases, $|E(A_2, B_1)| < b(2^i, k, |A_2| + |B_1|) + |A_2| + |B_1|$. Thus, we have $|E(G)| = |E(A_1, B_1)| + |E(A_2, B_1)| + |E(A, B_2)| < (b(2^i, k, |A_1| + |B_1|) + |A_1|) + (b(2^i, k, |A_2| + |B_1|) + |A_2| + |B_1|) + (b(2^i, k, |A| + |B_2|) + b(2^i, k, |B_2|)) \leq 4(2^{2i})k + 2n(2^i - 1) + n = 2^{2(i+1)}k + n(2^{i+1} - 1) = b(2^{i+1}, k, n)$, as required.
- $|E(A_1, B_1)| \geq b(2^i, k, |A_1| + |B_1|) + |A_1|$. In this case, we also have $|E(A_2, B_1)| \geq b(2^i, k, |A_2| + |B_1|) + |A_2|$. Let u be an arbitrary vertex in B_1 . Observe that $|E(A_1, B_1 \setminus \{u\})| \geq b(2^i, k, |A_1| + |B_1|)$ and $|E(A_2, B_1 \setminus \{u\})| \geq b(2^i, k, |A_2| + |B_1|)$. We also have $|E(A, B_2 \cup \{u\})| \geq b(2^i, k, |A| + |B_2 \cup \{u\}|) + b(2^i, k, |B_2 \cup \{u\}|)$. It is not possible that $|E(A_1, B_2 \cup \{u\})| < b(2^i, k, |A_1| + |B_2 \cup \{u\}|)$ and $|E(A_2, B_2 \cup \{u\})| < b(2^i, k, |A_2| + |B_2 \cup \{u\}|)$ as otherwise, $|E(A, B_2 \cup \{u\})| = |E(A_1, B_2 \cup \{u\})| + |E(A_2, B_2 \cup \{u\})| < b(2^i, k, |A_1| + |B_2 \cup \{u\}|) + b(2^i, k, |A_2| + |B_2 \cup \{u\}|) = b(2^i, k, |A| + |B_2 \cup \{u\}|) + b(2^i, k, |B_2 \cup \{u\}|)$. So, there exist disjoint vertex sets X and Y such that $|E(X)| \geq b(2^i, k, |X|)$ and $|E(Y)| \geq b(2^i, k, |Y|)$ (either $X = A_1 \cup B_1 \setminus \{u\}$ and $Y = A_2 \cup B_2 \cup \{u\}$, or $X = A_2 \cup B_1 \setminus \{u\}$ and $Y = A_1 \cup B_2 \cup \{u\}$). Thus, by taking a suitable 2^i -coloring of X and a suitable 2^i -coloring of Y , we have that $G \in (2^{i+1}, k)$ -LCP, a contradiction.

So we have proved that the claim also holds when $j = i + 1$, i.e. if $m \geq b(2^{i+1}, k, n)$ then $G \in (2^{i+1}, k)$ -LCP. \square

Lemma 6. *Let $f(c, k, n) = (2c - 1)ck + 2n(c - 1)$. For every nonnegative integer i and every graph G with n vertices, if $m \geq f(2^i, k, n)$ then $G \in (2^i, k)$ -LCP.*

Proof. We will prove the lemma by induction on i . For the base case, observe that any graph with at least $k = f(1, k, n)$ edges is in $(1, k)$ -LCP for every k and n . We now assume the claim holds for any j smaller than $i + 1$ and want to prove it for $i + 1$. Consider a graph G with n vertices such that $G \notin (2^{i+1}, k)$ -LCP and $|E(G)| \geq f(2^i, k, n)$.

We will show that there exists a set $A \subseteq V(G)$ such that $f(2^i, k, |A|) \leq |E(A)| \leq f(2^i, k, |A|) + |A|$ (and thus $G[A] \in (2^i, k)$ -LCP). We may construct the set A as follows: initially $A = \emptyset$ and while $|E(A)| < f(2^i, k, |A|)$, add

an arbitrary vertex of $V(G) \setminus A$ to A . Let u be the last added vertex; we have $f(2^i, k, |A|) \leq |E(A)| \leq |E(A \setminus \{u\})| + |A \setminus \{u\}| < f(2^i, k, |A \setminus \{u\}|) + |A \setminus \{u\}| < f(2^i, k, |A|) + |A|$.

Let $B = V(G) \setminus A$. If $G[B] \in (2^i, k)$ -LCP, then $G \in (2^{i+1}, k)$ -LCP, a contradiction. So $|E(B)| < f(2^i, k, |B|)$. Furthermore, $|E(A, B)| < b(2^{i+1}, k, n)$, as otherwise we are done by Lemma 5. Finally, $|E(G)| = |E(A)| + |E(B)| + |E(A, B)| < f(2^i, k, |A|) + f(2^i, k, |B|) + n + b(2^{i+1}, k, n) = f(2^{i+1}, k, n)$. The claim holds when $j = i + 1$, which completes the proof. \square

Theorem 2. *The (c, k) -LOAD COLORING Problem admits a kernel with less than $f(2c, k, 2ck) = 16c^2k - 6ck$ edges.*

Proof. By Theorem 1, we can get a kernel with less than $2ck$ vertices. Let c' be the minimum power of 2 such that $c \leq c'$. Observe that $c' < 2c$ and thus by Lemma 6 we get a kernel with $|E(G)| \leq f(c', k, 2ck) < f(2c, k, 2ck) = 16c^2k - 6ck$. \square

We now consider an approximation algorithm for the c -LOAD COLORING Problem: Given a graph G and integer c , we wish to determine the maximum k , denoted k_{opt} , for which $G \in (c, k)$ -LCP. We define the approximation ratio $r(c) = \frac{k_{opt}}{k}$, where k is the output of the approximation algorithm.

Let $K(c)k$ be an upper bound of the number of edges in a kernel for (c, k) -LOAD COLORING and let $P(c) = \prod_{i=1}^c \frac{K(i)}{i}$. For $c = 1$, we may assume that $K(1) = 1$ as $(1, k)$ -LOAD COLORING is trivially polynomial time solvable. Hence $P(1) = 1$. For $c \geq 2$, we have $K(c) = 16c^2 - 6c$.

Theorem 3. *There is a $2^{c-1}P(c)$ -approximation algorithm for c -LOAD COLORING.*

Proof. We prove the claim by induction on c . For $c = 1$, we have $P(1) = 1$. Assume the lemma is true for all $c' < c$.

Let G be an instance for c -LOAD COLORING with n vertices and m edges. We may assume that G has no isolated vertices. Clearly, $k_{opt} \leq \frac{m}{c}$. Consider $k = \lfloor \frac{m}{K(c)} \rfloor$.

If $k = 0$, then $m < K(c)$ and we can find k_{opt} in $O(1)$ time.

Now let $k > 0$. If $n \leq 2ck$, then by the proof of Theorem 2, since $m \geq K(c)k$, $G \in (c, k)$ -LCP. So we return k , and $\frac{k_{opt}}{k} \leq \frac{m}{ck} \leq \frac{K(c)(k+1)}{ck} \leq \frac{2K(c)}{c} \leq 2^{c-1}P(c)$.

If $n \geq 2ck$ and G is reduced for (c, k) -LCP, then by Theorem 1, $G \in (c, k)$ -LCP and we return k as above. If $n \geq 2ck$ and G is not reduced for (c, k) -LCP, we can use Lemma 3 to reduce (G, c) to (G', c') with $c' < c$. By induction we may find k' such that $k'_{opt} \leq 2^{c'-1}P(c')k'$, where k'_{opt} is the optimal solution for c' -LOAD COLORING on G' . Now consider three cases.

- $k' \geq k$. Then $G' \in (c', k)$ -LCP and so $G \in (c, k)$ -LCP. This is also a Yes-Instance case which leads to the same conclusion.

- $k'_{opt} \leq 2^{c'-1}P(c')k' < k$. Because $k'_{opt} + 1 \leq k$, an obstacle from $O_{c-c',k}$ is also an obstacle from $O_{c-c',k'_{opt}+1}$, therefore G' can be derived from G using a reduction rule for $(c, k'_{opt} + 1)$ -LCP. Since $G' \notin (c', k'_{opt} + 1)$ -LCP, $G \notin (c, k'_{opt} + 1)$ -LCP. Thus $k_{opt} = k'_{opt}$. The algorithm may output k' which satisfies $k_{opt} = k'_{opt} \leq 2^{c'-1}P(c')k' \leq 2^{c-1}P(c)k$.
- $k' < k \leq 2^{c'-1}P(c')k'$. The algorithm gives k' as an approximation of k_{opt} . Then $\frac{k_{opt}}{k'} \leq \frac{m}{ck'} \leq \frac{K(c)(k+1)}{ck'} \leq \frac{K(c)}{c} \frac{2k}{k'} \leq \frac{K(c)}{c} 2^{c'} P(c') \leq 2^{c-1}P(c)$.

In every case, the approximation ratio is at most $2^{c-1}P(c)$. \square

5 Number of Edges in Kernel for $c = 2$

In this section, we look into the edge kernel problem for the special case when $c = 2$. By doing a refined analysis, we will give a kernel with less than $8k$ edges for $(2, k)$ -LCP, which is a better bound than the general one. Henceforth, we assume that G is reduced for $(2, k)$ -LCP, and just consider the case when $|V(G)| < 4k$, as we have proved that if $|V(G)| \geq 4k$ then $G \in (2, k)$ -LCP.

Lemma 7. *If G has at least $3k - 2$ edges and every component in G has less than k edges then $G \in (2, k)$ -LCP.*

Proof. We consider colorings of the graph such that vertices in the same component are colored with the same color. Thus every edge in the graph is colored with 1 or 2. Denote the set of edges colored i with E_i , $i = 1, 2$. Among all possible colorings, choose a coloring of the graph such that $|E_1| \geq |E_2|$ and $||E_1| - |E_2||$ is minimum. Suppose $|E_2| \leq k - 1$, then $|E_1| \geq 2k - 1$, $||E_1| - |E_2|| > k$. Changing the color of one component from 1 to 2, we get a new coloring of the graph. For the new coloring, denote the set of edges colored i with E'_i , $i = 1, 2$. Since each component has less than k edges, $|E_1| > |E'_1| \geq k$, $|E'_2| \leq 2k - 2$. So $||E'_1| - |E'_2|| < ||E_1| - |E_2||$, a contradiction. Therefore we have $|E_1| \geq |E_2| \geq k$, so $G \in (2, k)$ -LCP. \square

If G has at least two components, each with at least k edges, it is obviously a Yes-instance. Therefore by Lemma 7, we may assume there is exactly one component C with at least k edges in the graph. Denote the total number of edges in $G - V(C)$ with m' . Observe that if $m' \geq k$, trivially $G \in (2, k)$ -LCP. So assume that $m' < k$.

Lemma 8. *If G is a reduced graph for $(2, k)$ -LCP, $m' < k$ and $\Delta = \Delta(G) \geq 3k - 2m'$, then $G \in (2, k)$ -LCP.*

Proof. Let u be one of the vertices with degree Δ and $N(u)$ its neighbors. Because the graph is reduced by Reduction Rule $R_{1,k}$, u has at least $2k - 2m'$ neighbors which are not leaves. Arbitrarily select $k - m'$ vertices among them and for each one, select any neighbor but u . Color the selected vertices and $G - V(C)$ by 1. By construction, there are at least k edges colored 1 and there

are at most $2k - 2m'$ colored vertices in $N(u)$. So there are at least k uncolored vertices in $N(u)$. We color them and u with 2. So $G \in (2, k)$ -LCP. \square

The next lemma deals with the case $\Delta = \Delta(G) < 3k$.

Lemma 9. *Let G be a graph with $\Delta < 3k$ and $|E(G)| \geq 8k$, then $G \in (2, k)$ -LCP.*

Proof. Because of Lemma 7, we may assume there exists a connected component C with at least k edges. In this component, choose a minimal set $A \subseteq V(C)$ such that $|A| \leq k + 1$ and $|E(A)| = k + d \geq k$. We may find such a set A in the following way. Select arbitrarily a vertex in C and put it into A , then keep adding to this set some neighbor of some vertex in A until $|E(A)| = k + d \geq k$. Since each time we select a neighbor of A we strictly increase $|E(A)|$, $|A| \leq k + 1$. If there is any vertex $u \in A$ with $|N_A(u)| \leq d$, then $A' = A \setminus \{u\}$ is a smaller vertex set such that $|E(A')| \geq k$. Thus, we may remove such vertices until $|E(A)| = k + d$ and for each vertex $u \in A$, $|N_A(u)| > d$. Denote $B = V(G) \setminus A$. We may assume $|E(B)| < k$, as otherwise $G \in (2, k)$ -LCP.

We now show that $|A| + d \leq k + 3$. Since every vertex $u \in A$ has $d_A(u) > d$, $|E(A)| = \frac{1}{2} \sum_{u \in A} d_A(u) \geq \frac{d+1}{2} |A|$. We have $k + d = |E(A)| \geq \frac{d+1}{2} |A|$, thus $|A| \leq \frac{2(k+d)}{d+1}$. Moreover as $d \leq |A| - 1$,

$$d + |A| \leq 2|A| - 1 \leq \frac{4(k+d)}{d+1} - 1 < \frac{4k}{d+1} + 3$$

If $d \geq 3$, we have our result, otherwise $d \leq 2$ and $d + |A| \leq 2 + k + 1 = k + 3$.

Let A_1, A_2, B_1, B_2 be a partition of $V(G)$ such that $A = A_1 \cup A_2$, $B = B_1 \cup B_2$, $|A_2| = 1$ and $|E(A, B_2)| < 2k$. Such a partition is possible: let $y = \arg \max\{|N_B(u)| : u \in A\}$ and initially take $A_1 = A \setminus \{y\}$, $A_2 = \{y\}$, $B_1 = B$, $B_2 = \emptyset$. Suppose $|E(A_1, B_1)| \leq k + |A_1|$ then $|E(G)| \leq |E(A)| + |E(B)| + |E(A_1, B_1)| + |E(A_2, B_1)| \leq (k + d) + (k - 1) + (k + |A| - 1) + \Delta \leq 7k + 1$, a contradiction since $|E(G)| \geq 8k$. So, $|E(A_1, B_1)| > k + |A_1|$. We will consider two cases: $\max\{|N_{B_1}(u)| : u \in A\}$ is greater than k or not.

If so, observe that $|E(A_2, B_1)| = |E(\{y\}, B_1)| = \max\{|N_{B_1}(u)| : u \in A\} > k$. Move all vertices of $B_1 \setminus N(y)$ to B_2 . We still have $|E(\{y\}, B_1)| > k$ and $|E(\{y\}, B_2)| = 0$. Moreover $B_1 \subseteq N(y)$. If $|E(A_1, B_2)| \geq k$, then G is in $(2, k)$ -LCP, thus $|E(A_1, B_2)| < k$. While $|E(\{y\}, B_1)| \geq k + 1$ and $|E(A_1, B_1)| \geq k + |A_1|$, move an arbitrary vertex from B_1 to B_2 . After each move, $|E(\{y\}, B_1)| \geq k$ and $|E(A_1, B_1)| \geq k$, thus $|E(A_2, B_2)| < k$ and $|E(A_1, B_2)| < k$ as otherwise, G would be in $(2, k)$ -LCP.

Eventually, we have $|E(A_1, B_1)| < k + |A_1|$ or $|E(\{y\}, B_1)| = k$. Suppose $|E(A_1, B_1)| < k + |A_1|$, then $|E(G)| \leq |E(A)| + |E(B)| + |E(A_1, B_1)| + |E(A_1, B_2)| + |E(\{y\}, B)| \leq (k + d) + (k - 1) + (k + |A_1| - 1) + (k - 1) + \Delta \leq 4k - 3 + (d + |A|) + \Delta < 8k$, a contradiction. Thus, $|E(A_1, B_1)| \geq k + |A_1|$ and $|E(\{y\}, B_1)| = k$. As $B_1 \subseteq N(y)$, we have $|B_1| = k$. We have found a new partition with the wanted properties and with $\max\{|N_{B_1}(u)| : u \in A\} \leq |B_1| = k$.

We now study the case $\max\{|N_{B_1}(u)| : u \in A\} \leq k$. While there exists $u \in B_1$ such that $|E(A, B_2 \cup \{u\})| < 2k$, move u from B_1 to B_2 . Then, (if and)

while $|E(A_1, B_1)| \geq k + |A_1|$ and $|E(A_2, B_1)| < k + |A_2|$, move an arbitrary vertex from A_1 to A_2 .

After all such moves, suppose that $|E(A_1, B_1)| < k + |A_1|$. If $|A_2| = 1$, we have $|E(A_2, B_1)| \leq \max\{|N_{B_1}(u)| : u \in A\} \leq k$, otherwise we moved some vertices from A_1 to A_2 . Let u be the last one. Since $|E(A_2 \setminus \{u\}, B_1)| < k + |A_2 \setminus \{u\}|$, we know $|E(A_2, B_1)| \leq |E(A_2 \setminus \{u\}, B_1)| + \max\{|N_{B_1}(u)| : u \in A\} < k + |A_2| - 1 + k = 2k + |A_2| - 1$. For both cases, $|E(G)| = |E(A)| + |E(B)| + |E(A_1, B_1)| + |E(A_2, B_1)| + |E(A, B_2)| \leq (k + d) + (k - 1) + (k + |A_1| - 1) + (2k + |A_2| - 2) + (2k - 1) \leq 7k + d + |A| - 5 < 8k$, which is impossible.

So, $|E(A_1, B_1)| \geq k + |A_1|$ which implies $|E(A_2, B_1)| \geq k + |A_2|$. For any vertex $u \in B_1$, we have $|E(A_1, B_1 \setminus \{u\})| \geq k$ and $|E(A_2, B_1 \setminus \{u\})| \geq k$ and we also obtain $|E(A, B_2 \cup \{u\})| \geq 2k$, i.e. $E(A_1, B_2 \cup \{u\})$ or $E(A_2, B_2 \cup \{u\})$ has at least k edges. Thus $G \in (2, k)$ -LCP. \square

The lemmas of this section and the fact that their proofs can be turned into polynomial algorithms, imply the following:

Theorem 4. *If G is reduced for $(2, k)$ -LCP and has at least $8k$ edges, then $G \in (2, k)$ -LCP. Thus, $(2, k)$ -LOAD COLORING admits a kernel with less than $8k$ edges.*

Since we have a better bound for the number of edges in a kernel when $c = 2$, we may get a better approximation when $c = 2$.

Theorem 5. *For every $\varepsilon > 0$, there is a $(4 + \varepsilon)$ -approximation algorithm for 2-LOAD COLORING.*

Proof. Let G be an instance for 2-LOAD COLORING with $m = 8p + q$ edges, where $0 \leq q < 8$. Let k_{opt} be the optimal solution of 2-LOAD COLORING on G , and observe that $k_{opt} \leq \lfloor \frac{m}{2} \rfloor \leq 4p + 3$. Let $p_0 = \lceil \frac{3}{\varepsilon} \rceil$. If $p \leq p_0 - 1$ then we can find k_{opt} in $O(1)$ time.

So assume that $p \geq p_0$. Note that $\frac{k_{opt}}{p} \leq \frac{4p+3}{p} \leq 4 + \varepsilon$. If G is reduced for $(2, p)$ -LCP, $G \in (2, p)$ -LCP by Theorem 4, and so p gives the required approximation. We may assume that G is not reduced for $(2, p)$ -LCP and reduce G to G' . If $|E(G')| \geq p$, then $G' \in (1, p)$ -LCP, and by Lemma 2, $G \in (2, p)$ -LCP. Again, p gives the required approximation.

Now assume that $|E(G')| < p$ and let $k'_{opt} = |E(G')|$ be the optimal solution of 1-LOAD COLORING on G' . Then $k'_{opt} + 1 \leq p$ and so an obstacle from $O_{1,p}$ is also an obstacle from $O_{1,k'_{opt}+1}$. Therefore, G' can be derived from G using a reduction rule for $(2, k'_{opt}+1)$ -LCP. Since $G' \notin (1, k'_{opt}+1)$ -LCP, $G \notin (2, k'_{opt}+1)$ -LCP. Thus $k_{opt} = k'_{opt} = |E(G')|$. So let our algorithm output $|E(G')|$ in this case. \square

6 Discussions

In the JUDICIOUS BIPARTITION Problem (see, e.g., the survey [9]), given a graph G , we are asked to find a bipartition V_1, V_2 of $V(G)$ which minimizes

$\max\{|E(V_1)|, |E(V_2)|\}$. To see that JUDICIOUS BIPARTITION and 2-LOAD COLORING are different problems, following [1] consider $2nK_2$, the union of $2n$ disjoint edges, and observe that while the solution of 2-LOAD COLORING is n , that of JUDICIOUS BIPARTITION is zero.

To the best of our knowledge, we obtained the first linear-edge kernel for a nontrivial problem on general graphs. As we could see, such kernels can be used to obtain approximation algorithms. It would be interesting to obtain such kernels for other nontrivial problems.

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